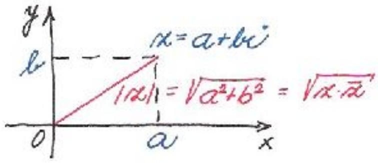
5. Absolutní hodnota komplexního čísla

př. $z \cdot \bar{z} = (a+bi)(a-bi) = a^2 - \underbrace{b^2 i^2}_{+b^2} = a^2+b^2$

$\sqrt{z \cdot \bar{z}} = \sqrt{a^2+b^2}$ $\quad$ ($\bar{z}$... kompl. sdruž. č.)



- ABSOLUTNÍ HODNOTA KOMPLEXNÍHO ČÍSLA $z = a+bi$ ... zn. $|z|$

  $|z| = \sqrt{a^2+b^2} = \sqrt{z \cdot \bar{z}}$

  - GEOMETRICKÝ VÝZNAM: vzdálenost obrazu kompl. čísla v Gaussově rovině od počátku soustavy souřadnic

- VĚTY:
  - pro $\forall z \in C$: $|z| \geq 0$ (pro $z=0$... $|z|=0$; pro $z \neq 0$... $|z|>0$)

    $|z| = |-z|$

  - pro $\forall z_1, z_2 \in C$: $|z_1 z_2| = |z_1| \cdot |z_2|$ $\quad$ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ pro $z_2 \neq 0$

př. $|3+2i| = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$

$\left|\frac{4-3i}{2+i}\right| = \frac{|4-3i|}{|2+i|} = \frac{\sqrt{16+(-3)^2}}{\sqrt{2^2+1^2}} = \frac{\sqrt{25}}{\sqrt{5}} = \frac{5}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \sqrt{5}$ $\quad$ [jinak $\frac{\sqrt{25}}{\sqrt{5}} = \sqrt{\frac{25}{5}} = \sqrt{5}$]

(1. zp.) s využ. podílu abs. hodnot $\quad$ [!, nedávat i "]

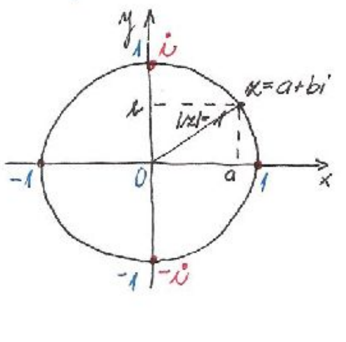
(2. zp.) $\left|\frac{4-3i}{2+i}\right| = \left|\frac{(4-3i)(2-i)}{(2+i)(2-i)}\right| = \left|\frac{8-4i-6i+\overbrace{3i^2}^{-3}}{4-\underbrace{i^2}_{+1}}\right| = \left|\frac{5-10i}{5}\right| = \left|\frac{5(1-2i)}{5}\right| =$

$= |1-2i| = \sqrt{1^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}$ $\quad$ [pozor na přesné zápisy, kdy $\sqrt{x}$, $|x|$, i]

- KOMPLEXNÍ JEDNOTKA: komplexní číslo, jehož absolutní hodnota rovna 1

  $|z| = 1$

  - je jich nekonečně mnoho, jejich obrazy leží v Gaussově rovině na KRUŽNICI se středem v počátku $O[0,0]$ a poloměrem 1



- REÁLNÉ (dvě): $[1,0] = 1$, $[-1,0] = -1$
- IMAGINÁRNÍ (dvě): $[0,1] = i$ $\quad$ $[0,-1] = -i$
- ostatní: $[a,b] = a+bi$, kde $\sqrt{a^2+b^2} = 1$

  např. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

  $|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4}+\frac{2}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$

Příklady

① Urči absolutní hodnotu

a) $z_1 = \dfrac{1+2i}{2-i} + 1 - 2i = \dfrac{1+2i}{2-i}\cdot\dfrac{2-i}{2-i} + 1 - 2i = \dfrac{2+i+4i+\overbrace{2i^2}^{-2}}{4-\underbrace{i^2}_{+1}} + 1 - 2i =$

upravíme na algeb. tvar

$= \dfrac{5i}{5} + 1 - 2i = i + 1 - 2i = \underset{Re(z_1)=1 \quad Im(z_1)=-1}{\underbrace{1}-i} \Rightarrow |z_1| = \sqrt{a^2+b^2} = \sqrt{1^2+(-1)^2} = \underline{\sqrt{2}}$

b) $z_2 = \dfrac{1+2i}{1+4i}$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

$|z_2| = \left|\dfrac{1+2i}{1+4i}\right| = \dfrac{|1+2i|}{|1+4i|} = \dfrac{\sqrt{1^2+2^2}}{\sqrt{1^2+4^2}} = \dfrac{\sqrt{5}}{\sqrt{17}}\cdot\dfrac{\sqrt{17}}{\sqrt{17}} = \underline{\dfrac{\sqrt{85}}{17}}$

pokud z_2 v podílu $\Rightarrow$ využ. podíl abs. hodnot

(2. zp.) z_2 na alg. tvar $\Rightarrow |z_2|$

$|z_2| = \left|\dfrac{1+2i}{1+4i}\cdot\dfrac{1-4i}{1-4i}\right| = \left|\dfrac{1-4i+2i-\overbrace{8i^2}^{+8}}{1-\underbrace{16i^2}_{+16}}\right| = \left|\dfrac{9-2i}{17}\right| = \left|\dfrac{9}{17}-\dfrac{2}{17}i\right| =$

$= \sqrt{\left(\dfrac{9}{17}\right)^2+\left(-\dfrac{2}{17}\right)^2} = \sqrt{\dfrac{81}{17^2}+\dfrac{4}{17^2}} = \sqrt{\dfrac{85}{17^2}} = \dfrac{\sqrt{85}}{17} \quad \left(\dfrac{\sqrt{85}}{\sqrt{17^2}}\right)$

POZOR NA SPRÁVNÉ ZÁPISY !!!

(nebo lze využít zápis a))

② Urči

a) $\left|1-i+\dfrac{1+2i}{3-i}\right| = \left|1-i+\dfrac{1+2i}{3-i}\cdot\dfrac{3+i}{3+i}\right| = \left|1-i+\dfrac{3+i+6i+\overbrace{2i^2}^{-2}}{9-\underbrace{i^2}_{+1}}\right| =$

(1. zp.) upr. na alg. tvar

$= \left|1-i+\dfrac{1+7i}{10}\right| = \left|\dfrac{10-10i+1+7i}{10}\right| = \left|\dfrac{11-3i}{10}\right| = \left|\underbrace{\dfrac{11}{10}-\dfrac{3}{10}i}_{\text{alg. tvar}}\right| = \sqrt{\left(\dfrac{11}{10}\right)^2+\left(-\dfrac{3}{10}\right)^2} =$

$= \sqrt{\dfrac{121+9}{100}} = \sqrt{\dfrac{130}{100}} = \dfrac{\sqrt{130}}{\sqrt{100}} = \underline{\dfrac{\sqrt{130}}{10}}$

$\sqrt{\dfrac{13}{10}} = \dfrac{\sqrt{13}}{\sqrt{10}}\cdot\dfrac{\sqrt{10}}{\sqrt{10}} = \dfrac{\sqrt{130}}{10}$ (množitorní)

(2. zp.) $\left|1-i+\dfrac{1+2i}{3-i}\right| = \left|\dfrac{(1-i)(3-i)+1+2i}{3-i}\right| = \left|\dfrac{3-i-3i+\overbrace{i^2}^{-1}+1+2i}{3-i}\right| = \left|\dfrac{3-2i}{3-i}\right| = \dfrac{|3-2i|}{|3-i|} =$

na společ. jmen. $\Rightarrow$ podíl 2 kompl. č. $\Rightarrow$ podíl abs. hodnot

$= \dfrac{\sqrt{9+4}}{\sqrt{9+1}} = \dfrac{\sqrt{13}}{\sqrt{10}}\cdot\dfrac{\sqrt{10}}{\sqrt{10}} = \underline{\dfrac{\sqrt{130}}{10}}$ [TENTO ZPŮSOB ASI VÝHODNĚJŠÍ]

b) $\big||2-3i|-i|2+i|\big| = \left|\sqrt{2^2+(-3)^2}-i\sqrt{2^2+1^2}\right| = \left|\sqrt{13}-i\sqrt{5}\right| = \sqrt{(\sqrt{13})^2+(-\sqrt{5})^2} =$

$= \sqrt{13+5} = \sqrt{18} = \sqrt{2\cdot 9} = \underline{3\sqrt{2}}$

c) DÚ $\left|\dfrac{i}{\sqrt{3}+i\sqrt{2}}\right| = \dfrac{|i|}{|\sqrt{3}+i\sqrt{2}|} = \dfrac{1}{\sqrt{(\sqrt{3})^2+(\sqrt{2})^2}} = \dfrac{1}{\sqrt{3+2}} = \dfrac{1}{\sqrt{5}}\cdot\dfrac{\sqrt{5}}{\sqrt{5}} = \underline{\dfrac{\sqrt{5}}{5}}$

podíl abs. h.

d) $|\,|2+3i|^2+(2+3i)^2| = |(\sqrt{4+9})^2+4+12i+\underbrace{9i^2}_{-9}| = |(\sqrt{13})^2-5+12i| =$

$= |13-5+12i| = |8+12i| = \sqrt{8^2+12^2} = \sqrt{64+144} = \sqrt{208} = \sqrt{4\cdot 52} = 2\sqrt{4\cdot 13} = \underline{4\sqrt{13}}$

③ Rozložte v součiny v C

a) $x^2+9y^2 = (x^2-9y^2i^2) = (x-3yi)(x+3yi)$

a^2-b^2 $[-i^2=1]$ $(a-b)(a+b)$

DÚ b) $4x^2+1 = 4x^2\underbrace{-i^2}_{+1} = (2x-i)(2x+i)$

DÚ c) $5+2y^2 = 5-2y^2i^2 = (\sqrt{5}-\sqrt{2}yi)(\sqrt{5}+\sqrt{2}yi)$

DÚ d) $1+y^2 = 1-y^2i^2 = (1-yi)(1+yi)$

④ Určete, která z čísel jsou komplexní jednotky

a) $z = \frac{\sqrt{3}-i}{2}$ $\quad |z| = \left|\frac{\sqrt{3}-i}{2}\right| = \frac{|\sqrt{3}-i|}{|2|} = \frac{\sqrt{(\sqrt{3})^2+1^2}}{2} = \frac{\sqrt{3+1}}{2} = \frac{\sqrt{4}}{2} = \frac{2}{2} = 1$

$?\,|z|=1$ $\quad$ z je kompl. jedn. $\quad$ $|2| = 2$ (jako v R)

DÚ b) $\frac{1+i}{2}$ $\quad \left|\frac{1+i}{2}\right| = \frac{|1+i|}{|2|} = \frac{\sqrt{1^2+1^2}}{2} = \frac{\sqrt{2}}{2}$ $\quad$ NENÍ

$\left[\ \left|\frac{1+i}{2}\right| = \left|\underbrace{\frac{1}{2}+\frac{1}{2}i}_{\text{alg. tvar}}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}\ \right]$

DÚ c) $\frac{(2+i)^2}{3-4i}$ $\quad \left|\frac{(2+i)^2}{3-4i}\right| = \frac{|(2+i)^2|}{|3-4i|} = \frac{|4+4i+\overset{-1}{i^2}|}{|3-4i|} = \frac{|3+4i|}{|3-4i|} = \frac{\sqrt{9+16}}{\sqrt{9+(-4)^2}} = \frac{5}{5} = 1$

JE KOMP. JEDN.

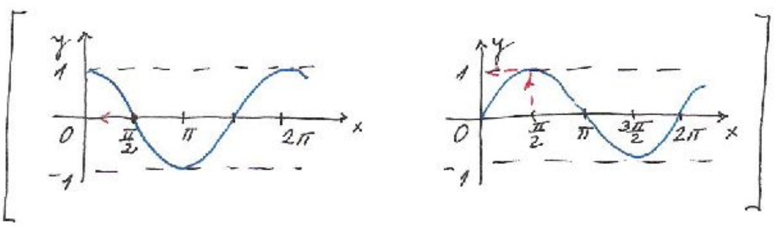
d) $-i^{30} = -i^{4\cdot 7+2} = -i^2 = -(-1) = 1$ $\quad$ JE

$[i^{4k+n} = i^n]$ $\quad k\in Z$

DÚ e) $\frac{1-i}{\sqrt{2}}$ $\quad \left|\frac{1-i}{\sqrt{2}}\right| = \frac{|1-i|}{|\sqrt{2}|} = \frac{\sqrt{1^2+(-1)^2}}{\sqrt{2}} = \frac{\sqrt{1+1}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$ $\quad$ JE

f) $\underbrace{\cos\frac{\pi}{2}}_{0} + i\underbrace{\sin\frac{\pi}{2}}_{1} = 0+1i = i$ $\quad |i| = \sqrt{1^2} = 1$ $\quad$ ANO

$[i = 0+1i]$



⑤ Řešte v C rovnice

pokud $|x| \Rightarrow$
$x = a+bi$
$|x| = \sqrt{a^2+b^2}$

a) $|x| - x = 1 + 2i$

ozn. $x = a+bi$ $\quad |x| = \sqrt{a^2+b^2}$ $\quad a, b \in R$

$\sqrt{a^2+b^2} - (a+bi) = 1+2i$

$\underbrace{\sqrt{a^2+b^2} - a}_{\text{reál. část}} \underbrace{- bi}_{Im} = \underbrace{1}_{Re} + \underbrace{2i}_{Im}$

z rovnosti kompl. čísel
(porovnáme reál. č., imag. č.)

$\sqrt{a^2+b^2} - a = 1$
$-b = 2 \Rightarrow b = -2$

$\sqrt{a^2+4} - a = 1$ IRACIONÁLNÍ RCE (součástí řeš. ZK)

$\sqrt{a^2+4} = 1 + a \quad /^2$

$a^2 + 4 = 1 + 2a + a^2$

$3 = 2a \quad /:2$

$a = \frac{3}{2}$

$x = a + bi = \frac{3}{2} - 2i$

Zkouška:

$L = \left|\frac{3}{2} - 2i\right| - \left(\frac{3}{2} - 2i\right) =$

$= \sqrt{\left(\frac{3}{2}\right)^2 + (-2)^2} - \frac{3}{2} + 2i =$

$= \sqrt{\frac{9}{4} + 4} - \frac{3}{2} + 2i =$

$= \sqrt{\frac{9+16}{4}} - \frac{3}{2} + 2i =$

$= \sqrt{\frac{25}{4}} - \frac{3}{2} + 2i = \frac{5}{2} - \frac{3}{2} + 2i =$

$= \frac{2}{2} + 2i = 1 + 2i$

$P = 1 + 2i \qquad L = P$

$K = \left\{\frac{3}{2} - 2i\right\}$

ROVNOST KOMPL. ČÍSEL (viz cv. 3)

$z_1 = a + bi$, $z_2 = c + di$

$z_1 = z_2 \Leftrightarrow \underbrace{a = c}_{\text{reál. č. se rovnají}} \wedge \underbrace{b = d}_{\text{imag. části se rovnají}}$

b) $x^2 + |x| = 0$

ozn. $x = a + bi$ $\quad |x| = \sqrt{a^2+b^2}$

$(a+bi)^2 + \sqrt{a^2+b^2} = 0$

$a^2 + 2abi + \underbrace{b^2i^2}_{-b^2} + \sqrt{a^2+b^2} = 0$

$\underbrace{a^2 - b^2 + \sqrt{a^2+b^2}}_{Re} \underbrace{+ 2abi}_{Im} = \underbrace{0}_{Re} + \underbrace{0i}_{Im}$

porovnáme Re, Im (bez „i")

$a^2 - b^2 + \sqrt{a^2+b^2} = 0 \quad (*)$

$2ab = 0 \Rightarrow a = 0 \vee b = 0$

(*) 1. $a = 0$ dos. do (*)

$-b^2 + \sqrt{b^2} = 0$

$\sqrt{b^2} = b^2 \quad /^2$

$b^2 = b^4$

$b^4 - b^2 = 0$

$b^2(b^2 - 1) = 0$

$[b \cdot b \cdot (b-1)(b+1) = 0]$

$b_1 = 0 \quad b_2 = 1 \quad b_3 = -1$

$x_1 = 0 + 0i = 0$

$x_2 = 0 + 1i = i$

$x_3 = 0 - 1i = -i$

2. $b = 0$ dos. do (*)

$a^2 + \sqrt{a^2} = 0$

$\sqrt{a^2} = -a^2$

$\Leftrightarrow a = 0$

$\left[\sqrt{a^2} = |a| \geq 0\right.$ pro $\forall a \in R$, tj. nemůže se rovnat $\underbrace{-a^2}_{\geq 0} \leq 0$, ? pouze pro $a = 0$ pl.$\left.\right]$

[kdyby $\sqrt{a^2} = -a^2 \quad /^2$

$a^2 = a^4$

$\vdots$

$a^2(a^2 - 1) = 0$

$a_1 = 0 \quad a_2 = 1 \quad a_3 = -1$

$\Rightarrow$ zk. pro $a_2 = 1$, $a_3 = -1$ nebude platit]

Zk:

$L(0) = 0^2 + 0 = 0$
$P(0) = 0 \quad L(0) = P(0)$

$L(i) = i^2 + |i| = -1 + 1 = 0$
$P(i) = 0 \quad L(i) = P(i)$

$L(-i) = (-i)^2 + |i| = -1 + 1 = 0$
$P(-i) = 0 \quad L(-i) = P(-i)$

$x_1 = 0 \quad x_2 = 0 + i = i \quad x_3 = 0 - i = -i$

$K = \{0, i, -i\}$

c) $\left(2-\frac{1}{i}\right)\bar{x} + 2x = 10i$

ozn. $x = a+bi \quad \bar{x} = a-bi$

$$\left(2-\frac{1}{i}\cdot\frac{-i}{-i}\right)(a-bi) + 2(a+bi) = 10i$$

$$\left(2-\frac{-i}{\underbrace{-i^2}_{1}}\right)(a-bi) + 2(a+bi) = 10i$$

$$(2+i)(a-bi) + 2(a+bi) = 10i$$

$$2a - 2bi + ai - \underbrace{bi^2}_{+b} + 2a + 2bi = 10i$$

$$\underbrace{4a+b}_{Re} + \underbrace{ai}_{Im} = 10i \quad (0+10i)$$

z rovnosti kompl. č. plyne

$$4a+b=0$$
$$a=10$$

$$40+b=0$$
$$b=-40$$

$$x = a+bi = 10-40i$$

$\mathcal{K} = \{10-40i\}$

d) $\bar{x}(x-1) = |x-1|^2$

ozn. $x = a+bi \quad \bar{x} = a-bi$

$$(a-bi)(a+bi-1) = |a+bi-1|^2$$

$$a^2 + \cancel{abi} - a - \cancel{abi} - \underbrace{b^2i^2}_{+b^2} + bi = |\underbrace{a-1}_{Re} \underbrace{+bi}_{Im}|^2$$

$$a^2+b^2-a+bi = \left(\sqrt{(a-1)^2+b^2}\right)^2$$

$$a^2+b^2-a+bi = (a-1)^2+b^2$$

$$\cancel{a^2}+\cancel{b^2}-a+bi = \cancel{a^2}-2a+1+\cancel{b^2}$$

$$a+bi = \underset{(1+0i)}{1}$$

z rovnosti kompl. čísel

$$a=1$$
$$b=0$$

$$x = 1+0i = 1$$

$\mathcal{K} = \{1\}$